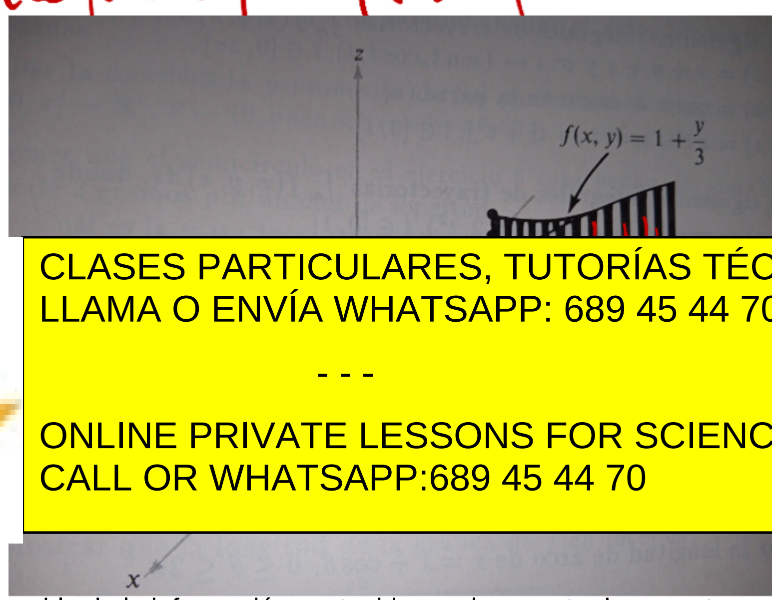
Example on line integrals for scalar functions:

The aunt of Tom Sawyer asked him to paint both sides of her fence.

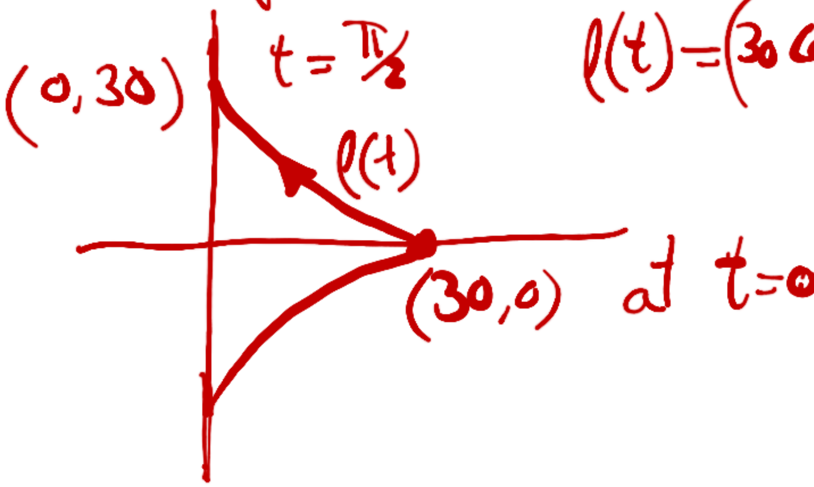
He thinks she should pay 5 cents every 25 feet^2

How much can Tom earn with this work under those conditions?

Fence is given by a function $f(x,y) = 1 + \frac{y}{3}$ along a trajectory $r(t) = (30\cos^3 t,\ 30\sin^3 t)$



We have $\ell(t)$, $I = [0, \frac{\pi}{2}]$ just considering the first quadrant.



$$\ell(t) = (30\cos^3 t,\ 30\sin^3 t)$$

We might take a partition of $[a,b]$

$$a = t_0 < \dots < t_k = b$$

$$\Delta t = \frac{b-a}{k}$$

Adding all of those areas

$$A = \sum_{j=1}^{k} f(\rho(t_j)) \cdot \Delta S_j = \sum_{j=1}^{k} f(\rho(t_j)) \, \|\rho'(t)\| \Delta t$$

mean value theorem.

$\Delta S_j \to 0$

$$\int_a^b f(\rho(t)) \, \|\rho'(t)\| \, dt$$

$$f(x,y) = 1 + \frac{y}{3}$$

$$\rho(t) = (30\cos^3 t, 30\sin^3 t) \quad t \in [0, \tfrac{\pi}{2}]$$

$$f(\rho(t)) = 1 + 10\sin^3 t$$

$$\int_0^{\pi/2} \left(1+10\sin^3 t\right) 90 \sin t \cos t \, dt$$

$$= 90 \int_0^{\pi/2} \left(\underline{\sin t} \cos t + 10 \underline{\sin^4 t} \cos t\right) dt$$

$$= 90 \left(\frac{\sin^2 t}{2} + 2\sin^5 t \right) \Bigg]_0^{\pi/2} = 90 \left(\frac{1}{2} + 2 \right)$$

$$= \frac{450}{2} = 225$$ (one side of the fence in the first quadrant.)

Total area $= 4 \cdot 225 = \underline{900}$ feet^2

Path integrals for vector fields

$\sigma : [a,b] \to \mathbb{R}^N$ trajectory.

$F : \mathbb{R}^N \to \mathbb{R}^N$ vector field.

We would like to compute the line of F along $\sigma([a,b]) \equiv$ curve.

$$F : \sigma([a,b]) \to \mathbb{R}^N$$

Definition

scalar product.

$$\int_\sigma F = \int_a^b \underbrace{F(\sigma(t))}_{\text{vector}} \cdot \underbrace{\sigma'(t)}_{\text{vector}} \, dt$$

Example: Find the path integral of

$$F(x,y,z) = (e^y, e^x, e^z)$$

along $\sigma(t) = (0, t, t^2)$ $t \in [0, \log 2]$

$$\int_\sigma F = \int_0^{\log 2} F(\sigma(t)) \cdot \sigma'(t)\, dt$$

$$F(\sigma(t)) = (e^t, 1, e^{t^2})$$

$$\sigma'(t) = (0, 1, 2t)$$

$$\int_\sigma F = \int_0^{\log 2} \left(1 + 2t e^{t^2}\right) dt = \left(t + e^{t^2}\right)_0^{\log 2}$$

Relation between line integrals for scalar fields and vector fields

$\sigma : [a,b] \to \mathbb{R}^n$ trajectory.

$$\int_\sigma \mathbf{F}\cdot d\mathbf{s} = \boxed{\int_a^b F(\sigma(t))\cdot \sigma'(t)\,dt}$$

scalar product

$$= \int_a^b \underbrace{\left(F(\sigma(t)) \cdot \frac{\sigma'(t)}{\|\sigma'(t)\|} \right)}_{g(\sigma(t))} \cdot \|\sigma'(t)\|\,dt$$

multiplication of numbers

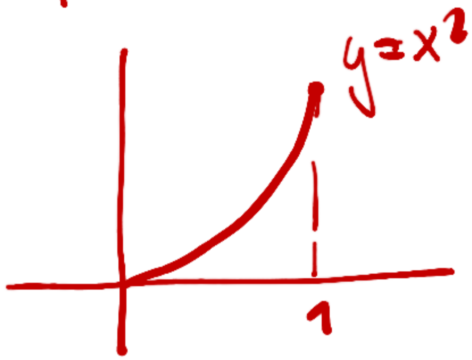
$g(\sigma(t))$ scalar function.

$$= \int^b g(\sigma(t)) \cdot \|\sigma'(t)\|\,dt$$

Parametrizations

We do not have a unique parametrization.

Example: $\{(t,t^2),\ t\in[0,1]\}$



If $\left.\begin{array}{l} x=t \\ y=t^2 \end{array}\right\}\ t\in[0,1]$

$$\{(2t,(2t)^2),\ t\in[0,\tfrac{1}{2}]\}$$

Both represent the same curve.

[illegible] with the line integral depending

Proposition

Two parametrizations of the same curve σ, ρ

$f: \mathbb{R}^n \to \mathbb{R}$ scalar field.

Then,
$$\int_\sigma f = \int_\rho f$$

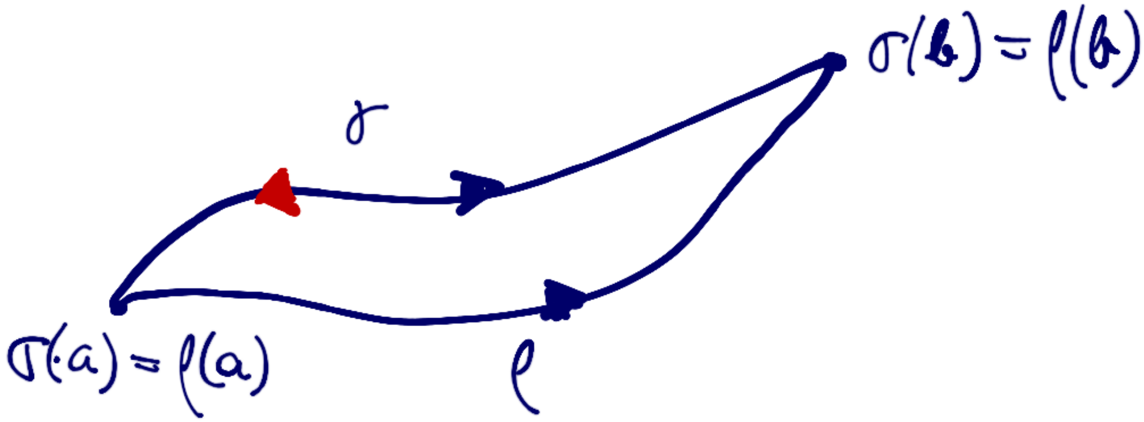
Proposition

Two paramt. σ and ρ. $F: \mathbb{R}^n \to \mathbb{R}^n$ vector field.

a) If σ and ρ have the same orientation

$$\int F = \int F$$

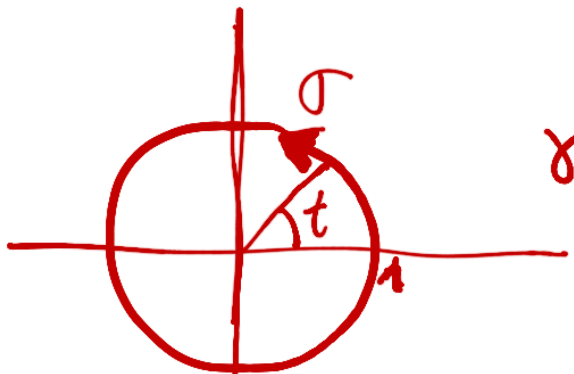
$$\int_\sigma \quad \int_\rho$$



Example: Circle $\{x^2+y^2=1\}$

$$\sigma : [0, 2\pi] \to \mathbb{R}^2$$
$$t \to (\cos t, \sin t) = \sigma(t)$$

anticlockwise $\equiv$ positive orientation



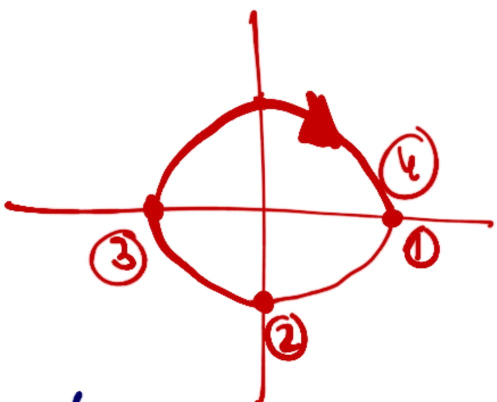
$$\gamma : [0, 2\pi] \to \mathbb{R}^2$$
$$t \to \gamma(t) = (\cos(2\pi - t), \sin(2\pi - t))$$

$$\text{at } t=0 \quad \gamma(0)=(1,0)$$

$$\text{at } t=\frac{\pi}{2} \quad \gamma\left(\frac{\pi}{2}\right)=(0,-1)$$

$$\text{at } t=\pi \quad \gamma(\pi)=(-1,0)$$

$$\text{at } t=2\pi \quad \gamma(2\pi)=(1,0)$$



$$\int_\sigma f = \int_a^b f(\sigma(t)) \cdot \underline{\|\sigma'(t)\|}\, dt$$

$F(\sigma(t))$

Fundamental Theorem of Calculus

$\sigma : [a,b] \longrightarrow \mathbb{R}^N$ trajectory , $\sigma \in C^1$

and

$f : \sigma([a,b]) \longrightarrow \mathbb{R}^N$ scalar field along σ

$f \in C^1$

Then,

$$\int_\sigma \nabla f \cdot ds = f(\sigma(b)) - f(\sigma(a))$$

Remark

· Important because $\nabla f \equiv$ vector field.

So we can define

σ

Example: We would like to compute

$$\int_\sigma \mathbf{F}\cdot d\mathbf{s} = \int_\sigma \overbrace{y\,dx + x\,dy}^{\mathbf{F}\cdot d\mathbf{s}} \quad \text{with } \sigma(t) = \left(t^9, \sin^9\left(\frac{\pi t}{2}\right)\right)$$

$$t \in [0,1]$$

$F(x,y) = (y, x)$ vector field.

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\int_\sigma F\cdot ds = \int_0^1 F(\sigma(t))\cdot \sigma'(t)\,dt = \int_0^1 \left(9t^8 \underline{\sin^9\left(\frac{\pi t}{2}\right)} + \cdots\right)$$

$$\begin{cases} F(\sigma(t)) = \left(\underline{\sin^9\left(\frac{\pi t}{2}\right)}, t^9\right) \\ \sigma'(t) = \left(9t^8, 9\sin^8\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)\cdot\frac{\pi}{2}\right) \end{cases}$$

Looking the vector field.

$$\boxed{F(x,y) = \nabla f(x,y)} \quad \text{with} \quad f(x,y) = xy$$

$$F(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$$

Applying FTC.

FTC.

$$\int_\sigma F \cdot ds = \boxed{\int_\sigma \nabla f \cdot ds = f(\sigma(1)) - f(\sigma(0))}$$

$$\boxed{1}$$

$$\int_0^1 \nabla f(\sigma(t)) \cdot \sigma'(t)\, dt$$

In 1D

$$\int f(x)\,dx = F(x)+C$$ antiderivative.

$$\underline{F'(x)=f(x)}$$

$$\Downarrow$$

$$\int F'(x)\,dx$$

Barrow's law

$$\int_a^b F'(x)\,dx = F(b)-F(a)$$

Definition

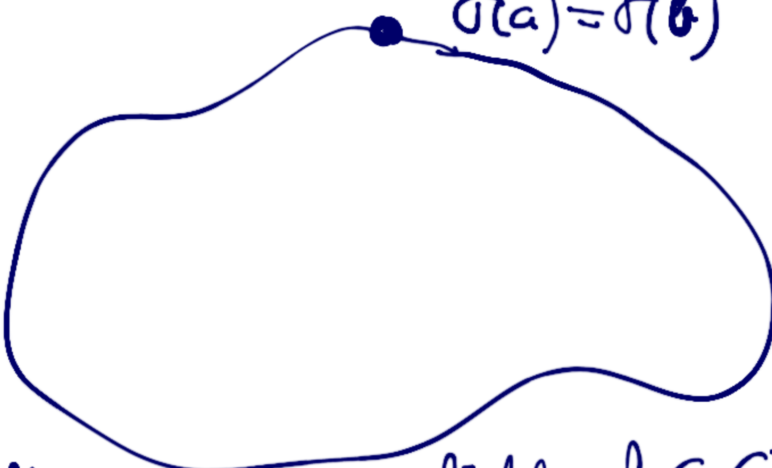
If $F = \nabla f$ then we say that F is a conservative field

Corollary (consequence)

$\sigma : [a, b] \longrightarrow \mathbb{R}^n$ trajectory of a closed curve

$\sigma(a) = \sigma(b)$ $\quad \sigma([a, b])$

$\sigma \in C^1$



$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ scalar field, $f \in C^1$

Then

Definition

$F : \Omega \longrightarrow \mathbb{R}^N$ vector field and $\Omega \subset \mathbb{R}^N$

with F having a $\underbrace{\text{exact differential form}}_{\text{conservative field.}}$

Then, there exists a scalar field

$$f : \mathbb{R}^N \longrightarrow \mathbb{R}.$$

such that $\nabla f = F$ in Ω

f is called a potential function.

Ex: F force.

Theorem

D is an open set simply connected
- any closed curve can be deformed continuously inside D
- In $\mathbb{R}^2, \mathbb{R}^3$ this means there are no holes.

$F \in C^1(D)$ vector field.

Then the following is equivalent:

a) F is conservative $\nabla f = F$

b) For any closed curve $\int_\sigma F = 0$

c) If we have two different parametrizations

d) In $\mathbb{R}^2$ if $F=(P,Q)$

then $\boxed{\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x} \sim \dfrac{\partial F_1}{\partial y}=\dfrac{\partial F_2}{\partial x}}$

since $F=\left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)=(F_1, F_2)$

then we have.

$$\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial^2 f}{\partial x \partial y} \quad \text{Schwarz Th.}$$

e) In $\mathbb{R}^3$

$\operatorname{curl} F = \operatorname{rot} F = 0$ equivalent to d) in $\mathbb{R}^2$

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

If we were in $\mathbb{R}^2$, $F_3 = 0$ for

$$F = (F_1, F_2, F_3) = (F_1, F_2, 0) \text{ and}$$

$$F_1 = F_1(x,y)$$
$$F_2 = F_2(x,y)$$

then

Problem 1 – set 4.2

$$F(x,y,z) = \left(\sin y + z,\ x\cos y + e^z,\ x + ye^z\right)$$

a) Show that F is conservative.

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z & x\cos y + e^z & x + ye^z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}\left(x+ye^z\right) - \frac{\partial}{\partial z}\left(x\cos y + e^z\right)\right)$$

$$= i\left(e^z - e^z\right) - j\left(1 - 1\right) + k\left(\cos y - \cos y\right)$$

$$= 0 \quad (\text{vector})$$

$$\Downarrow$$

F is conservative $\Longrightarrow$ $\exists\, \phi$ such that $F = \nabla \phi$

b) Compute $\phi : \mathbb{R}^3 \to \mathbb{R}$.

$$F = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$= \left(\sin y + z, \; x \cos y + e^z, \; x + y e^z \right)$$

$$\frac{\partial \phi}{\partial z} = x + y e^z$$

$$\frac{\partial \phi}{\partial x} = \sin y + z \Rightarrow \phi(x,y,z) = \int (\sin y + z)\, dx$$

$$\boxed{\phi(x,y,z) = \boxed{x\sin y} + \boxed{xz} + A(y,z)}$$

$$\frac{\partial \phi}{\partial y} = x\cos y + e^{z} \Rightarrow \phi(x,y,z) = \int (x\cos y + e^{z})\, dy$$

$$\boxed{\phi(x,y,z) = \boxed{x\sin y} + \boxed{ye^{z}} + B(x,z)}$$

$$\frac{\partial \phi}{\partial z} = x + ye^{z} \Rightarrow \phi(x,y,z) = \int (x + ye^{z})\, dz$$

$$\phi(x,y,z) = \boxed{xz} + \boxed{ye^{z}} + \boxed{C(x,y)}$$